Spectral Method for Analysis of Switching Diffusions

K. A. Rybakov and I. L. Sotskova

Abstract—**The analysis problem of switching diffusions is considered. This paper presents a new approach based on the spectral method formalism for solving generalized Fokker-Planck equations. The proposed method allows to transform partial differential equations into the linear algebraic equations, and to arrive at a solution in an explicit form. The aspects of applications are discussed. A numerical example is given to illustrate the efficiency of the proposed method.**

Index Terms—**Switching diffusions, generalized Fokker-Planck equations, spectral method, spectral transform.**

I. Introduction

We consider models of complicated control systems that arise in numerous applications such as navigation and flight control of an aircraft [1], pancake landing in the turbulent atmosphere [2], fault-tolerant control systems [3], flexible manufacturing systems [4], Markowitz's mean-variance portfolio selection with regime switching [5], etc.

The system state is given by a pair $(X(t),K(t)) \in \mathbb{R}^n \times \{1,2,\ldots,N\}$, where X and K are continuous and discrete components, respectively; $t \in T$, $T=[t_0,t_1]$. The evolution of the process $(X(t),K(t))$ is described by the following equations:

$$\begin{aligned} dX(t) &= f(t,X(t),K(t))dt \\ &\quad + \sigma(t,X(t),K(t))dW(t), \quad X(t_0)=X_0, \end{aligned} \tag{1}$$

$$\begin{aligned} &\mathrm{P}(K(t+\Delta t)=r \mid K(t)=k, X(t)=x) \\ &\quad = \lambda_{kr}(t,x)\Delta t + o(\Delta t), \quad K(t_0)=K_0, \end{aligned} \tag{2}$$

where $f^{\langle k\rangle}(t,x)=f(t,x,k)\colon T\times\mathbb{R}^n \to \mathbb{R}^n$ is an n-dimensional continuous function, $\sigma^{\langle k\rangle}(t,x)=\sigma(t,x,k)\colon T\times\mathbb{R}^n\to\mathbb{R}^{n\times s}$ is an $(n\times s)$-dimensional continuous function, $\lambda_{kr}(t,x)\colon T\times\mathbb{R}^n\to[0,+\infty)$ is a continuous intensity function, $W(t)$ is an s-dimensional Wiener process independent of X_0 and K_0; $k,r=1,2,\ldots,N$, $k\neq r$.

Assumption 1.1:

(i) There exists $C_1,C_2>0$ such that $|f^{\langle k\rangle}(t,x)|+\|\sigma^{\langle k\rangle}(t,x)\| < C_1(1+|x|)$, $|f^{\langle k\rangle}(t,x)-f^{\langle k\rangle}(t,y)|+\|\sigma^{\langle k\rangle}(t,x)-\sigma^{\langle k\rangle}(t,y)\| + \|\lambda(t,x)-\lambda(t,y)\| < C_2|x-y|$ for any $t\in T$, $x,y\in\mathbb{R}^n$, $k=1,2,\ldots,N$. Here $\lambda(t,x)$ is the $(N\times N)$-dimensional function with entries $\lambda_{kr}(t,x)$ ($\lambda_{kk}(t,x)\equiv 0$).

(ii) $\mathbb{E}[|X_0|^2]<\infty$ ($\mathbb{E}[\,\cdot\,]$ is the expectation).

For any $t\in T$ the most comprehensive statistical characteristic of a pair $(X(t),K(t))$ is the function $\phi(t,x,k)\colon T\times\mathbb{R}^n\times\{1,2,\ldots,N\}\to[0,+\infty)$, where $\phi^{\langle k\rangle}(t,x)=\phi(t,x,k)\colon T\times\mathbb{R}^n\to[0,+\infty)$ is the nonnormalized probability density of $X(t)$ when $K(t)=k$; $k=1,2,\ldots,N$. Thus, $\mathrm{P}^{\langle k\rangle}(t)=\int_{\mathbb{R}^n}\phi^{\langle k\rangle}(t,x)dx$ is the probability that $K(t)=k$, $\varphi(t,x)=\sum_{k=1}^N\phi^{\langle k\rangle}(t,x)$ is the probability density of $X(t)$, and $\tilde\phi^{\langle k\rangle}(t,x)=[\mathrm{P}^{\langle k\rangle}(t)]^{-1}\phi^{\langle k\rangle}(t,x)$ is the conditional probability density of $X(t)$ when $K(t)=k$. It is known that if $\phi^{\langle 1\rangle}(t,x)$, …, $\phi^{\langle N\rangle}(t,x)$ exist then they satisfy generalized Fokker-Planck equations [1], [3], [6], [7]:

$$\begin{aligned} \frac{\partial\phi^{\langle k\rangle}(t,x)}{\partial t} &= -\sum_{i=1}^n\frac{\partial}{\partial x_i}\left[f_i^{\langle k\rangle}(t,x)\phi^{\langle k\rangle}(t,x)\right] \\ &\quad + \frac12\sum_{i,j=1}^n\frac{\partial^2}{\partial x_i\partial x_j}\left[g_{ij}^{\langle k\rangle}(t,x)\phi^{\langle k\rangle}(t,x)\right] \\ &\quad - \sum_{r=1,r\neq k}^N\lambda_{kr}(t,x)\phi^{\langle k\rangle}(t,x) + \sum_{r=1,r\neq k}^N\lambda_{rk}(t,x)\phi^{\langle r\rangle}(t,x), \end{aligned} \tag{3}$$

$$\phi^{\langle k\rangle}(t_0,x)=\phi_0^{\langle k\rangle}(x), \quad \phi^{\langle k\rangle}(t,x)\Big|_{x=\pm\infty}=0, \ k=1,2,\ldots,N.$$

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Here $g^{\langle k\rangle}(t,x)=\sigma^{\langle k\rangle}(t,x)[\sigma^{\langle k\rangle}(t,x)]'\colon T\times\mathbb{R}^n\to\mathbb{R}^{n\times n}$ is the diffusion matrix ($[\,\cdot\,]'$ denotes the transpose), $\phi_0^{\langle k\rangle}(x)$ is the given nonnormalized probability density of X_0 when $K_0=k$, i.e.,

$$\phi_0^{\langle k\rangle}(x)=\phi(t_0,x,k), \qquad \sum_{k=1}^N\int_{\mathbb{R}^n}\phi_0^{\langle k\rangle}(x)dx=1.$$

So, the analysis problem of switching diffusions (1), (2) is formulated as follows: Given functions $f^{\langle k\rangle}(t,x)$ and $\sigma^{\langle k\rangle}(t,x)$ defining Itô equation (1), intensities $\lambda_{kr}(t,x)$ characterizing the process $K(t)$, and nonnormalized probability densities $\phi_0^{\langle k\rangle}(x)$ of X_0, find nonnormalized probability densities $\phi^{\langle k\rangle}(t,x)$; $k,r=1,2,\ldots,N$, $k\neq r$.

Remark 1.1: The condition (i) of Assumption 1.1 can be weakened [1], [8], [9].

Assumption 1.2:

(i) There exists the probability density $\varphi(t,x)$ (there exists $\phi^{\langle k\rangle}(t,x)$, $k=1,2,\ldots,N$).

(ii) $\phi^{\langle k\rangle}(t,x),\varphi(t,x)\in L_2(T\times\mathbb{R}^n)$; $\phi_0^{\langle k\rangle}(x)\in L_2(\mathbb{R}^n)$;

(iii) For any $\xi(t,x)\in C_0^\infty(T\times\mathbb{R}^n)$ the following equations are satisfied ($k=1,2,\ldots,N$):

$$\begin{aligned} &\int_T\int_{\mathbb{R}^n}\xi(t,x)\frac{\partial\phi^{\langle k\rangle}(t,x)}{\partial t}dxdt \\ &\quad = \sum_{i=1}^n\int_T\int_{\mathbb{R}^n}\frac{\partial\xi(t,x)}{\partial x_i}f_i^{\langle k\rangle}(t,x)\phi^{\langle k\rangle}(t,x)dxdt \\ &\qquad + \frac12\sum_{i,j=1}^n\int_T\int_{\mathbb{R}^n}\frac{\partial^2\xi(t,x)}{\partial x_i\partial x_j}g_{ij}^{\langle k\rangle}(t,x)\phi^{\langle k\rangle}(t,x)dxdt \\ &\qquad - \sum_{r=1,r\neq k}^N\int_T\int_{\mathbb{R}^n}\xi(t,x)\lambda_{kr}(t,x)\phi^{\langle k\rangle}(t,x)dxdt \\ &\qquad + \sum_{r=1,r\neq k}^N\int_T\int_{\mathbb{R}^n}\xi(t,x)\lambda_{rk}(t,x)\phi^{\langle r\rangle}(t,x)dxdt, \end{aligned}$$

where $C_0^\infty(T\times\mathbb{R}^n)$ consists of functions that have the compact support and continuous derivatives of all orders.

Methods for a solution of generalized Fokker-Planck equations are similar to methods for a solution of the classical Fokker-Planck equation [10], [11], and therefore they have the same advantages and imperfections. The simplest method is the Gaussian approximation; however, it is the least accurate, since this method does not give the exact solution even for the linear control systems in contrast to stochastic systems without regime switching.

Methods based on the representation of the probability density by series on orthogonal functions have obtained the basic distribution [1]. These methods allow to pass from generalized Fokker-Planck equations to a large order system of ordinary differential equations, but its solving demands the significant time expenses. The numerical solving generalized Fokker-Planck equations by using the methods developed for the classical Fokker-Planck equation and the Monte Carlo method underlie other approaches to the analysis problem [12]. The detailed description of different methods for solving generalized Fokker-Planck equations is given in [1].

A new approach based on the spectral method formalism [13], [14], [15] to solve the analysis problem of switching diffusions is given in this paper. The proposed method allows to transform generalized Fokker-Planck equations into the linear algebraic equations, and to arrive at a solution in an explicit form.

II. Preliminary results

Assume that $n_1 = n+1$. Let $\{q_{i_0}(t)\}_{i_0=0}^{\infty}$ be an orthonormal basis of $L_2(T)$ and let $\{p_{i_1\dots i_n}(x)\}_{i_1,\dots,i_n=0}^{\infty}$ be an orthonormal basis of $L_2(\mathbb{R}^n)$, then $\{e_{i_0\dots i_n}(t,x)\}_{i_0,\dots,i_n=0}^{\infty}$ is the orthonormal basis of $L_2(T\times\mathbb{R}^n)$, where $e_{i_0\dots i_n}(t,x) = q_{i_0}(t)p_{i_1\dots i_n}(x)$.

Definition 2.1: An infinite n_1-dimensional matrix $H(n_1,0) = [h_{i_0\dots i_n}]$ is called the spectral characteristic of a function $h(t,x) \in L_2(T\times\mathbb{R}^n)$ if $h_{i_0\dots i_n} = (e_{i_0\dots i_n}(t,x), h(t,x))_{L_2(T\times\mathbb{R}^n)}$.

Thus, $H(n_1,0) = \mathbb{S}\,[h(t,x)]$ if and only if

$$h_{i_0\dots i_n} = \int_T\int_{\mathbb{R}^n} e_{i_0\dots i_n}(t,x)h(t,x)dxdt,\ i_0,\dots,i_n = 0,1,2,\dots,$$

then

$$\begin{aligned} h(t,x) &= \mathbb{S}^{-1}\,[H(n_1,0)] \\ &= \sum_{i_0,\dots,i_n=0}^{\infty} h_{i_0\dots i_n} e_{i_0\dots i_n}(t,x), \quad (t,x)\in T\times\mathbb{R}^n, \end{aligned} \tag{4}$$

where $\mathbb{S}$ and $\mathbb{S}^{-1}$ denote the spectral transform and the spectral inversion, respectively.

Similarly, the spectral characteristic of a function $h(x) \in L_2(\mathbb{R}^n)$ may be defined.

Definition 2.2: An infinite n-dimensional matrix $H(n,0) = [h_{i_1\dots i_n}]$ is called the spectral characteristic of a function $h(x) \in L_2(\mathbb{R}^n)$ if

$$\begin{aligned} h_{i_1\dots i_n} &= (p_{i_1\dots i_n}(x), h(x))_{L_2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} p_{i_1\dots i_n}(x)h(x)dx, \quad i_1,\dots,i_n = 0,1,2,\dots\ . \end{aligned}$$

Definition 2.3: An infinite $2n_1$-dimensional matrix $A(n_1,n_1) = [a_{i_0\dots i_n j_0\dots j_n}]$ is said to be the spectral characteristic of a linear operator $\mathcal{A}\colon D_{\mathcal{A}} \subseteq L_2(T\times\mathbb{R}^n) \to L_2(T\times\mathbb{R}^n)$ if

$$\begin{aligned} a_{i_0\dots i_n j_0\dots j_n} &= (e_{i_0\dots i_n}(t,x), \mathcal{A}e_{j_0\dots j_n}(t,x))_{L_2(T\times\mathbb{R}^n)} \\ &= \int_T\int_{\mathbb{R}^n} e_{i_0\dots i_n}(t,x)\mathcal{A}e_{j_0\dots j_n}(t,x)dxdt, \\ &\quad i_0,\dots,i_n,j_0,\dots,j_n = 0,1,2,\dots\ . \end{aligned}$$

Before proceeding further, we present some preliminary results with regard to properties of the spectral characteristics (multidimensional matrix operations are described in Appendix).

Proposition 2.1: For any $h_l(t,x) \in L_2(T\times\mathbb{R}^n)$ and $\gamma_l \in \mathbb{R}$ ($l = 1,2,\dots,L$) the following equation is satisfied:

$$\mathbb{S}\left[\sum_{l=1}^{L}\gamma_l h_l(t,x)\right] = \sum_{l=1}^{L}\gamma_l\,\mathbb{S}\,[h_l(t,x)].$$

Proof: This follows from scalar product properties. ∎

Theorem 2.1: Let $h(t,x) \in L_2(T\times\mathbb{R}^n)$ be a function such that $h(t^*,x) = h^*(x) \in L_2(\mathbb{R}^n)$, and let $q(1,0;t^*)$ be the infinite column vector with entries $q_{i_0}(t^*)$, i.e., $q(1,0;t^*) = [q_0(t^*), q_1(t^*), q_2(t^*),\dots]'$; $t^*\in T$. Denote the spectral characteristic of $h^*(x)$ by $H^*(n,0)$. Then $([q(1,0;t^*)]'\otimes E(n,n))\cdot H(n_1,0) = H^*(n,0)$, where $E(n,n)$ is the $2n$-dimensional identity matrix.

Proof: For almost all $x\in\mathbb{R}^n$ the function $h(t,x)$ can be represented in the form

$$h(t,x) = \sum_{i_0=0}^{\infty} h_{i_0}(x)q_{i_0}(t),$$

where $h_{i_0}(x) = (q_{i_0}(t), h(t,x))_{L_2(T)} = \int_T q_{i_0}(t)h(t,x)dt$.

Due to Definition 2.2 the entries of $H^*(n,0)$ are defined by $h^*_{i_1\dots i_n} = (p_{i_1\dots i_n}(x), h^*(x))_{L_2(\mathbb{R}^n)}$, and therefore

$$\begin{aligned} h^*_{i_1\dots i_n} &= \left(p_{i_1\dots i_n}(x), \sum_{i_0=0}^{\infty} h_{i_0}(x)q_{i_0}(t^*)\right)_{L_2(\mathbb{R}^n)} \\ &= \sum_{i_0=0}^{\infty} q_{i_0}(t^*)(p_{i_1\dots i_n}(x), h_{i_0}(x))_{L_2(\mathbb{R}^n)} = \sum_{i_0=0}^{\infty} q_{i_0}(t^*)h_{i_0\dots i_n}, \end{aligned}$$

where $h_{i_0\dots i_n}$ are entries of the spectral characteristic of $h(t,x)$.

Let $Q^*(n,n_1) = [q(1,0;t^*)]'\otimes E(n,n)$ be the infinite $(n+n_1)$-dimensional matrix with entries defined by

$$q^*_{i_1\dots i_n j_0\dots j_n} = \begin{cases} q_{j_0}(t^*), & i_1 = j_1,\ \dots,\ i_n = j_n, \\ 0, & \text{otherwise}, \end{cases}$$

then

$$h^*_{i_1\dots i_n} = \sum_{j_0,\dots,j_n=0}^{\infty} q^*_{i_1\dots i_n j_0\dots j_n}h_{j_0\dots j_n}.$$

This implies that $Q^*(n,n_1)\cdot H(n_1,0) = H^*(n,0)$. Thus, we obtain the desired result. ∎

Theorem 2.2: Suppose that $\mathcal{A}\colon D_{\mathcal{A}} \subseteq L_2(T\times\mathbb{R}^n) \to L_2(T\times\mathbb{R}^n)$ is a linear operator, and $h(t,x)\in D_{\mathcal{A}}$. Let $A(n_1,n_1)$ be the spectral characteristic of $\mathcal{A}$. Then $\mathbb{S}\,[\mathcal{A}h(t,x)] = A(n_1,n_1)\cdot\mathbb{S}\,[h(t,x)]$.

Proof: Assume that $w(t,x) = \mathcal{A}h(t,x)$. Then, using (4), we have

$$\begin{aligned} w(t,x) &= \mathcal{A}\left[\sum_{i_0,\dots,i_n=0}^{\infty} h_{i_0\dots i_n}e_{i_0\dots i_n}(t,x)\right] \\ &= \sum_{i_0,\dots,i_n=0}^{\infty} h_{i_0\dots i_n}\mathcal{A}e_{i_0\dots i_n}(t,x), \end{aligned}$$

where $h_{i_0\dots i_n}$ are defined by Definition 2.1.

Let $W(n_1,0) = [w_{i_0\dots i_n}]$ be the spectral characteristic of $w(t,x)$, consequently,

$$\begin{aligned} w_{i_0\dots i_n} &= (e_{i_0\dots i_n}(t,x), w(t,x))_{L_2(T\times\mathbb{R}^n)} \\ &= \left(e_{i_0\dots i_n}(t,x), \sum_{j_0,\dots,j_n=0}^{\infty} h_{j_0\dots j_n}\mathcal{A}e_{j_0\dots j_n}(t,x)\right)_{L_2(T\times\mathbb{R}^n)} \\ &= \sum_{j_0,\dots,j_n=0}^{\infty}(e_{i_0\dots i_n}(t,x), \mathcal{A}e_{j_0\dots j_n}(t,x))_{L_2(T\times\mathbb{R}^n)}h_{j_0\dots j_n} \\ &= \sum_{j_0,\dots,j_n=0}^{\infty} a_{i_0\dots i_n j_0\dots j_n}h_{j_0\dots j_n}, \end{aligned}$$

where $a_{i_0\dots i_n j_0\dots j_n}$ are entries of $A(n_1,n_1)$ due to Definition 2.3. Therefore $W(n_1,0) = \mathbb{S}\,[\mathcal{A}h(t,x)] = A(n_1,n_1)\cdot\mathbb{S}\,[h(t,x)]$. ∎

Theorem 2.3: Suppose that $\mathcal{A}\colon D_{\mathcal{A}} \subseteq L_2(T\times\mathbb{R}^n) \to L_2(T\times\mathbb{R}^n)$ and $\mathcal{B}\colon D_{\mathcal{B}} \subseteq L_2(T\times\mathbb{R}^n) \to R_{\mathcal{B}} \subseteq L_2(T\times\mathbb{R}^n)$ are linear operators, $R_{\mathcal{B}} \subseteq D_{\mathcal{A}}$; $\mathcal{C} = \mathcal{A}\circ\mathcal{B}$ is a composition of $\mathcal{A}$ and $\mathcal{B}$. Denote spectral characteristics of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ by $A(n_1,n_1)$, $B(n_1,n_1)$, and $C(n_1,n_1)$, respectively. Then $C(n_1,n_1) = A(n_1,n_1)\cdot B(n_1,n_1)$.

Proof: Let $h(t,x) \in D_{\mathcal{B}}$. By virtue of Theorem 2.2, $\mathbb{S}\,[\mathcal{C}h(t,x)] = C(n_1,n_1)\cdot\mathbb{S}\,[h(t,x)]$. On the other hand, $\mathbb{S}\,[\mathcal{C}h(t,x)] = \mathbb{S}\,[\mathcal{A}[\mathcal{B}h(t,x)]] = A(n_1,n_1)\cdot\mathbb{S}\,[\mathcal{B}h(t,x)] = A(n_1,n_1)\cdot B(n_1,n_1)\cdot\mathbb{S}\,[h(t,x)]$. This implies that $C(n_1,n_1) = A(n_1,n_1)\cdot B(n_1,n_1)$, since $h(t,x)$ was arbitrary. ∎

Remark 2.1: Spectral characteristics with a similar properties can be defined for elements, which do not belong to $L_2(T\times\mathbb{R}^n)$ (for instance, for elements of $L_p(T\times\mathbb{R}^n)$, where $p < 2$) and for distributions [15], [16].

III. Spectral method for solving generalized Fokker-Planck equations

Apply the spectral transform to left-hand and right-hand sides of (3) by using the linearity (see Proposition 2.1) and remark 2.1. Then

$$\mathbb{S}\left[\frac{\partial\phi^{\langle k\rangle}(t,x)}{\partial t}\right] = -\sum_{i=1}^{n}\mathbb{S}\left[\frac{\partial}{\partial x_i}\left[f_i^{\langle k\rangle}(t,x)\phi^{\langle k\rangle}(t,x)\right]\right] + \frac{1}{2}\sum_{i,j=1}^{n}\mathbb{S}\left[\frac{\partial^2}{\partial x_i\partial x_j}\left[g_{ij}^{\langle k\rangle}(t,x)\phi^{\langle k\rangle}(t,x)\right]\right] - \sum_{r=1,r\neq k}^{N}\mathbb{S}\left[\lambda_{kr}(t,x)\phi^{\langle k\rangle}(t,x)\right] + \sum_{r=1,r\neq k}^{N}\mathbb{S}\left[\lambda_{rk}(t,x)\phi^{\langle r\rangle}(t,x)\right]. \quad (5)$$

We will use the following notations ($i,j = 1,2,\dots,n$; $k,r = 1,2,\dots,N$; $k\neq r$):

(i) $\mathcal{P}(n_1,n_1)$ is the spectral characteristic of the differentiation operator $\partial/\partial t$;

(ii) $\mathcal{P}_i(n_1,n_1)$ and $\mathcal{P}_{ij}(n_1,n_1)$ are spectral characteristics of the differentiation operators $\partial/\partial x_i$ and $\partial^2/\partial x_i\partial x_j$, respectively;

(iii) $F_i^{\langle k\rangle}(n_1,n_1)$, $G_{ij}^{\langle k\rangle}(n_1,n_1)$, and $\Lambda_{kr}(n_1,n_1)$ are spectral characteristics of the multiplication operators with multipliers $f_i^{\langle k\rangle}(t,x)$, $g_{ij}^{\langle k\rangle}(t,x)$, and $\lambda_{kr}(t,x)$, respectively.

Remark 3.1: The analytic expressions for spectral characteristics of the differential and multiplication operators for different orthonormal bases such as Legendre polynomials, Fourier basis, Walsh and Haar functions, Hermite functions are given in [13], [15].

Proposition 3.1: Let $\Phi^{\langle k\rangle}(n_1,0)$ and $\Phi_0^{\langle k\rangle}(n,0)$ be spectral characteristics of the nonnormalized probability densities $\phi^{\langle k\rangle}(t,x)$ and $\phi_0^{\langle k\rangle}(x)$, respectively; $k=1,2,\dots,N$. Then

(i) $\mathbb{S}\left[\frac{\partial\phi^{\langle k\rangle}(t,x)}{\partial t}\right] = P(n_1,n_1)\cdot\Phi^{\langle k\rangle}(n_1,0) - q(1,0;t_0)\otimes\Phi_0^{\langle k\rangle}(n,0)$, where $P(n_1,n_1) = \mathcal{P}(n_1,n_1) + (q(1,0;t_0)\cdot[q(1,0;t_0)]')\otimes E(n,n)$;

(ii) $\mathbb{S}\left[\frac{\partial}{\partial x_i}\left[f_i^{\langle k\rangle}(t,x)\phi^{\langle k\rangle}(t,x)\right]\right] = \mathcal{P}_i(n_1,n_1)\cdot F_i^{\langle k\rangle}(n_1,n_1)\cdot\Phi^{\langle k\rangle}(n_1,0)$;

(iii) $\mathbb{S}\left[\frac{\partial^2}{\partial x_i\partial x_j}\left[g_{ij}^{\langle k\rangle}(t,x)\phi^{\langle k\rangle}(t,x)\right]\right] = \mathcal{P}_{ij}(n_1,n_1)\cdot G_{ij}^{\langle k\rangle}(n_1,n_1)\cdot\Phi^{\langle k\rangle}(n_1,0)$;

(iv) $\mathbb{S}\left[\lambda_{kr}(t,x)\phi^{\langle k\rangle}(t,x)\right] = \Lambda_{kr}(n_1,n_1)\cdot\Phi^{\langle k\rangle}(n_1,0)$;

for each $i,j=1,2,\dots,n$ and $k,r=1,2,\dots,N$, $k\neq r$.

Proof: It is clear from Theorem 2.2 that

$$\mathbb{S}\left[\frac{\partial\phi^{\langle k\rangle}(t,x)}{\partial t}\right] = \mathcal{P}(n_1,n_1)\cdot\Phi^{\langle k\rangle}(n_1,0).$$

Representing the spectral characteristic $\mathcal{P}(n_1,n_1)$ as $\mathcal{P}(n_1,n_1) = P(n_1,n_1) - (q(1,0;t_0)\cdot[q(1,0;t_0)]')\otimes E(n,n)$ and using Theorem 2.1, we obtain that $\mathcal{P}(n_1,n_1)\cdot\Phi^{\langle k\rangle}(n_1,0) = (P(n_1,n_1) - (q(1,0;t_0)\cdot[q(1,0;t_0)]')\otimes E(n,n))\cdot\Phi^{\langle k\rangle}(n_1,0) = P(n_1,n_1)\cdot\Phi^{\langle k\rangle}(n_1,0) - (q(1,0;t_0)\otimes E(n,n))\cdot([q(1,0;t_0)]'\otimes E(n,n))\cdot\Phi^{\langle k\rangle}(n_1,0) = P(n_1,n_1)\cdot\Phi^{\langle k\rangle}(n_1,0) - (q(1,0;t_0)\otimes E(n,n))\cdot\Phi_0^{\langle k\rangle}(n,0) = P(n_1,n_1)\cdot\Phi^{\langle k\rangle}(n_1,0) - q(1,0;t_0)\otimes\Phi_0^{\langle k\rangle}(n,0)$.

The proof of (ii)–(iv) follows from Theorems 2.2 and 2.3. ∎

By virtue of Proposition 3.1, equations (5) can be rewritten in the form

$$P(n_1,n_1)\cdot\Phi^{\langle k\rangle}(n_1,0) - q(1,0;t_0)\otimes\Phi_0^{\langle k\rangle}(n,0) = -\sum_{i=1}^{n}\mathcal{P}_i(n_1,n_1)\cdot F_i^{\langle k\rangle}(n_1,n_1)\cdot\Phi^{\langle k\rangle}(n_1,0) + \frac{1}{2}\sum_{i,j=1}^{n}\mathcal{P}_{ij}(n_1,n_1)\cdot G_{ij}^{\langle k\rangle}(n_1,n_1)\cdot\Phi^{\langle k\rangle}(n_1,0) - \sum_{r=1,r\neq k}^{N}\Lambda_{kr}(n_1,n_1)\cdot\Phi^{\langle k\rangle}(n_1,0) + \sum_{r=1,r\neq k}^{N}\Lambda_{rk}(n_1,n_1)\cdot\Phi^{\langle r\rangle}(n_1,0), \quad k=1,2,\dots,N,$$

consequently, spectral characteristics $\Phi^{\langle 1\rangle}(n_1,0)$, $\dots$, $\Phi^{\langle N\rangle}(n_1,0)$ satisfy the following system of equations:

$$\begin{cases} P(n_1,n_1)\cdot\Phi^{\langle 1\rangle}(n_1,0) - A_{11}(n_1,n_1)\cdot\Phi^{\langle 1\rangle}(n_1,0) - \dots \\ -A_{1N}(n_1,n_1)\cdot\Phi^{\langle N\rangle}(n_1,0) = q(1,0;t_0)\otimes\Phi_0^{\langle 1\rangle}(n,0), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ P(n_1,n_1)\cdot\Phi^{\langle N\rangle}(n_1,0) - A_{N1}(n_1,n_1)\cdot\Phi^{\langle 1\rangle}(n_1,0) - \dots \\ -A_{NN}(n_1,n_1)\cdot\Phi^{\langle N\rangle}(n_1,0) = q(1,0;t_0)\otimes\Phi_0^{\langle N\rangle}(n,0), \end{cases} \quad (6)$$

where

$$A_{kk}(n_1,n_1) = -\sum_{i=1}^{n}\mathcal{P}_i(n_1,n_1)\cdot F_i^{\langle k\rangle}(n_1,n_1) + \frac{1}{2}\sum_{i,j=1}^{n}\mathcal{P}_{ij}(n_1,n_1)\cdot G_{ij}^{\langle k\rangle}(n_1,n_1) - \sum_{r=1,r\neq k}^{N}\Lambda_{kr}(n_1,n_1),$$

$$A_{kr}(n_1,n_1) = \Lambda_{rk}(n_1,n_1), \quad k,r=1,2,\dots,N, \quad k\neq r.$$

Thus, the analysis problem of switching diffusions (1), (2) is reduced to solving the infinite system of the linear algebraic equations (6) with the unknown entries of spectral characteristic $\Phi^{\langle 1\rangle}(n_1,0)$, $\dots$, $\Phi^{\langle N\rangle}(n_1,0)$. The aspects of solving the infinite system of the linear algebraic equations are given in [17].

Let $n_2 = n+2$ and $Z(n_2,n_2)$ be the $2n_2$-dimensional matrix such that

$$Z(n_2,n_2) = \begin{bmatrix} P(n_1,n_1) - A_{11}(n_1,n_1)\dots & -A_{1N}(n_1,n_1) \\ \vdots & \ddots & \vdots \\ -A_{N1}(n_1,n_1) & \dots P(n_1,n_1) - A_{NN}(n_1,n_1) \end{bmatrix},$$

and let $\Phi(n_2,0)$ and $B(n_2,0)$ be n_2-dimensional matrices such that

$$\Phi(n_2,0) = \begin{bmatrix} \Phi^{\langle 1\rangle}(n_1,0) \\ \vdots \\ \Phi^{\langle N\rangle}(n_1,0) \end{bmatrix},$$

$$B(n_2,0) = \begin{bmatrix} q(1,0;t_0)\otimes\Phi_0^{\langle 1\rangle}(n,0) \\ \vdots \\ q(1,0;t_0)\otimes\Phi_0^{\langle N\rangle}(n,0) \end{bmatrix},$$

then (6) takes the following form:

$$Z(n_2,n_2)\cdot\Phi(n_2,0) = B(n_2,0), \quad (7)$$

and therefore $\Phi(n_2,0) = Z^{-1}(n_2,n_2)\cdot B(n_2,0)$.

Remark 3.2: Matrices $P(n_1,n_1)$ and $A_{kr}(n_1,n_1)$ are infinite $2n_1$-dimensional matrices, $k,r = 1,2,\dots,N$; however, the $2n_2$-dimensional matrix $Z(n_2,n_2) = [z_{\alpha i_0\dots i_n\beta j_0\dots j_n}]$ and n_2-dimensional matrices $\Phi(n_2,0) = [\phi_{\alpha i_0\dots i_n}]$ and $B(n_2,0) = [b_{\alpha i_0\dots i_n}]$ are such that $i_0,\dots,i_n,j_0,\dots,j_n = 0,1,2,\dots$ and $\alpha,\beta = 1,2,\dots,N$. In order to determine $Z(n_2,n_2)$, $\Phi(n_2,0)$, and $B(n_2,0)$ it is necessary to use the matrix aggregation [13].

Spectral characteristics $\Phi^{\langle k\rangle}(n_1,0)$ can be derived from $\Phi(n_2,0)$ by the matrix decomposition [13], where $\Phi(n_2,0)$ satisfies (7). Then, the nonnormalized probability density $\phi^{\langle k\rangle}(t,x)$ is given by (4):

$$\phi^{\langle k\rangle}(t,x) = \mathbb{S}^{-1}\left[\Phi^{\langle k\rangle}(n_1,0)\right] = \sum_{i_0,\dots,i_n=0}^{\infty} \phi^{\langle k\rangle}_{i_0\dots i_n} e_{i_0\dots i_n}(t,x), \quad (t,x)\in T\times\mathbb{R}^n,$$

where $\phi^{\langle k\rangle}_{i_0\dots i_n}$ are entries of $\Phi^{\langle k\rangle}(n_1,0)$; $k=1,2,\dots,N$.

Remark 3.3: To find an approximate solution of the analysis problem of switching diffusions all spectral characteristics must be truncated on all dimensions that allow the infinite values of indexes (see Remark 3.2). The methodical inaccuracy caused by the spectral characteristic truncation is described in [14], [15].

IV. Analysis of the satellite stabilization

Consider a stabilization problem of the pico-satellite [18]. Suppose that the satellite dynamics in the orbital plane can be defined by the following linearized equations:

$$\begin{aligned} dX_1(t) &= X_2(t)dt, \\ X_1(0) &= X_{10}, \\ dX_2(t) &= (-X_1(t) + (2-K(t))u(t))dt + 0.1dW(t), \\ X_2(0) &= X_{20}, \end{aligned}$$

where X_1 is the deflection angle of the gravity gradient boom, X_2 is the angular velocity of the self-rotation, $t\in T=[0,1]$, $W(t)$ is a one-dimensional Wiener process, X_{10} and X_{20} are independent random variables. Thus, $n=2$, $N=2$ ($K\in\{1,2\}$); $f^{\langle k\rangle}(t,x_1,x_2) = [x_2, -x_1+(2-k)u(t)]'$, $\sigma^{\langle k\rangle}(t,x_1,x_2) = [0,0.1]'$; $k=1,2$. Consequently, this control system can work in two regimes: normal regime ($k=1$) and control fault ($k=2$).

Let $\phi^{\langle 1\rangle}_0(x_1,x_2) = \frac{1}{2\pi}\exp\left(-\frac{1}{2}\left([x_1+0.3]^2+[x_2-0.1]^2\right)\right)$ and $\phi^{\langle 2\rangle}_0(x_1,x_2)=0$ are nonnormalized probability densities of $[X_{10},X_{20}]'$, i.e., $\mathrm{P}^{\langle 1\rangle}(0)=1$, $\mathrm{P}^{\langle 2\rangle}(0)=0$, $\mathbb{E}[X_{10}]=-0.3$, $\mathbb{E}[X_{20}]=0.1$.

The control process $u(t) = -0.2(1-\tan 1-2\tan^2 1)\cos t - 0.2(3+3\tan 1+2\tan^2 1)\sin t$ guarantees that $\mathbb{E}[X_1(1)] = \mathbb{E}[X_2(1)] = 0$ if $\lambda_{12}(t,x_1,x_2)=\lambda_{21}(t,x_1,x_2)=0$ (there is no regime switching).

It is necessary to find nonnormalized probability densities $\phi^{\langle 1\rangle}(t,x_1,x_2)$, $\phi^{\langle 2\rangle}(t,x_1,x_2)$, probabilities $\mathrm{P}^{\langle 1\rangle}(t)$, $\mathrm{P}^{\langle 2\rangle}(t)$, and expectations $m_1(t)=\mathbb{E}[X_1(t)]$, $m_2(t)=\mathbb{E}[X_2(t)]$, when $\lambda_{12}(t,x_1,x_2)=0.2$, $\lambda_{21}(t,x_1,x_2)=0.1$. Hence the process $K(t)$ is the continuous time Markov chain with generator [4]:

$$\begin{bmatrix} -0.2 & 0.2 \\ 0.1 & -0.1 \end{bmatrix}.$$

Let $\{q_{i_0}(t) = \hat{P}_{i_0}(t)\}_{i_0=0}^{\infty}$ and $\{p_{i_1 i_2}(x_1,x_2) = \hat{H}_{i_1}(x_1)\,\hat{H}_{i_2}(x_2)\}_{i_1,i_2=0}^{\infty}$ be the orthonormal bases, where $\{\hat{P}_{i_0}(t)\}_{i_0=0}^{\infty}$ is the orthonormal Legendre polynomials, $\{\hat{H}_{i_l}(x_l)\}_{i_l=0}^{\infty}$ is the orthonormal Hermite functions; $l=1,2$. The approximate solution of the analysis problem is presented on Fig. 1 (see Remark 3.3; $i_0,i_1,i_2=0,1,\dots,7$).

It is known that functions $\mathrm{P}^{\langle k\rangle}(t)$ and $m_i(t)$ ($k=1,2$, $i=1,2$) are formally defined as

$$\mathrm{P}^{\langle k\rangle}(t) = \int_{\mathbb{R}^2}\phi^{\langle k\rangle}(t,x_1,x_2)dx_1dx_2,$$
$$m_i(t) = \int_{\mathbb{R}^2} x_i[\phi^{\langle 1\rangle}(t,x_1,x_2)+\phi^{\langle 2\rangle}(t,x_1,x_2)]dx_1dx_2. \quad (8)$$

Moreover, these functions can be found by the two-moment parametric approximation [1]. Then

$$\mathrm{P}^{\langle 1\rangle}(t) = \frac{1}{3}+\frac{2}{3}e^{-0.3t}, \quad \mathrm{P}^{\langle 2\rangle}(t) = \frac{2}{3}-\frac{2}{3}e^{-0.3t},$$
$$\begin{aligned} m_1(t) &= (0.5+0.18t-1.58e^{-0.3t})\sin t \\ &\quad + (2.25+0.42t-2.55e^{-0.3t})\cos t, \\ m_2(t) &= (-2.07-0.42t+3.02e^{-0.3t})\sin t \\ &\quad + (0.92+0.18t-0.82e^{-0.3t})\cos t. \end{aligned} \quad (9)$$

The graphs of probabilities $\mathrm{P}^{\langle 1\rangle}(t)$, $\mathrm{P}^{\langle 2\rangle}(t)$ and expectations $m_1(t)$, $m_2(t)$ are presented on Fig. 2 and 3, respectively. The exact solutions (9) are shown by dot lines, and the approximate solutions defined as in (8) are shown by solid lines. The satellite stabilization error is evaluated as $\mathbb{E}[X_1(1)]\approx 0.007$, $\mathbb{E}[X_2(1)]\approx 0.06$.

In this case $\Delta[\mathrm{P}^{\langle 1\rangle}]\approx 0.007337$, $\Delta[\mathrm{P}^{\langle 2\rangle}]\approx 0.001039$, $\Delta[m_1]\approx 0.005693$, $\Delta[m_2]\approx 0.008628$, where $\Delta[\mathrm{P}^{\langle k\rangle}] = \|\mathrm{P}^{\langle k\rangle}_{\mathrm{exact}}(t) - \mathrm{P}^{\langle k\rangle}_{\mathrm{approximate}}(t)\|_{L_2(T)}$ (the definition of $\Delta[m_i]$ is similar). It is seen that the spectral method is sufficiently effective; however, the accuracy can be increased. For example, if $i_0=0,1,\dots,7$ and $i_1,i_2=0,1,\dots,11$ then $\Delta[\mathrm{P}^{\langle 1\rangle}]\approx 0.000584$, $\Delta[\mathrm{P}^{\langle 2\rangle}]\approx 0.000080$, $\Delta[m_1]\approx 0.000728$, $\Delta[m_2]\approx 0.001076$; if $i_0=0,1,\dots,7$ and $i_1,i_2=0,1,\dots,15$ then $\Delta[\mathrm{P}^{\langle 1\rangle}]\approx 0.000046$, $\Delta[\mathrm{P}^{\langle 2\rangle}]\approx 0.000006$, $\Delta[m_1]\approx 0.000086$, $\Delta[m_2]\approx 0.000121$.

V. Conclusion

The proposed method for the analysis problem of switching diffusions is the effective implement for the construction of regular approximate solution methods and algorithms for the stochastic control systems. The efficiency and accuracy of the suggested method is demonstrated by the example given above.

It is possible to transform partial differential equations into the linear algebraic equations by means of the spectral characteristic calculation upon each variable separately and also to obtain their tensor products [13], [14]. This approach allows to increase the efficiency of computations.

Appendix
Multidimensional matrix operations

1. Let $\alpha,\beta\in\mathbb{R}$ and let $A(p,q)=[a_{i_1\dots i_p j_1\dots j_q}]$ and $B(p,q)=[b_{i_1\dots i_p j_1\dots j_q}]$ be the infinite $(p+q)$-dimensional matrices. The expression $\alpha A(p,q)+\beta B(p,q)$ is the infinite $(p+q)$-dimensional matrix $C(p,q)=[c_{i_1\dots i_p j_1\dots j_q}]$ if $c_{i_1\dots i_p j_1\dots j_q} = \alpha a_{i_1\dots i_p j_1\dots j_q} + \beta b_{i_1\dots i_p j_1\dots j_q}$, $i_1,\dots,i_p,j_1,\dots,j_q=0,1,2,\dots$.

2. Let $A(p,r)=[a_{i_1\dots i_p k_1\dots k_r}]$ and $B(r,q)=[b_{k_1\dots k_r j_1\dots j_q}]$ be the infinite $(p+r)$-dimensional and $(r+q)$-dimensional matrices, respectively. The product $A(p,r)\cdot B(r,q)$ is the infinite $(p+q)$-dimensional matrix $C(p,q)=[c_{i_1\dots i_p j_1\dots j_q}]$ if

$$c_{i_1\dots i_p j_1\dots j_q} = \sum_{k_1,\dots,k_r=0}^{\infty} a_{i_1\dots i_p k_1\dots k_r} b_{k_1\dots k_r j_1\dots j_q} < \infty,$$
$$i_1,\dots,i_p,j_1,\dots,j_q=0,1,2,\dots .$$

An infinite $2p$-dimensional matrix $E(p,p)$ is said to be the identity matrix if $A(p,p)\cdot E(p,p)=E(p,p)\cdot A(p,p)=A(p,p)$ for each $2p$-dimensional matrix $A(p,p)$.

3. Let $A(p,p)$ be an infinite $2p$-dimensional matrix. An infinite $2p$-dimensional matrix $B(p,p)$ is said to be the two-sided inverse of $A(p,p)$ if $A(p,p)\cdot B(p,p)=B(p,p)\cdot A(p,p)=E(p,p)$. We will use the notation $A^{-1}(p,p)$ to denote the two-sided inverse of $A(p,p)$.

4. Let $A(p,q)=[a_{i_1\dots i_p j_1\dots j_q}]$ and $B(r,s)=[b_{k_1\dots k_r l_1\dots l_s}]$ be the infinite $(p+q)$-dimensional and $(r+s)$-dimensional matrices, respectively. The tensor product $A(p,q)\otimes B(r,s)$ is the infinite $(p+r+q+s)$-dimensional matrix $C(p+r,q+s) = [c_{i_1\dots i_p k_1\dots k_r j_1\dots j_q l_1\dots l_s}]$

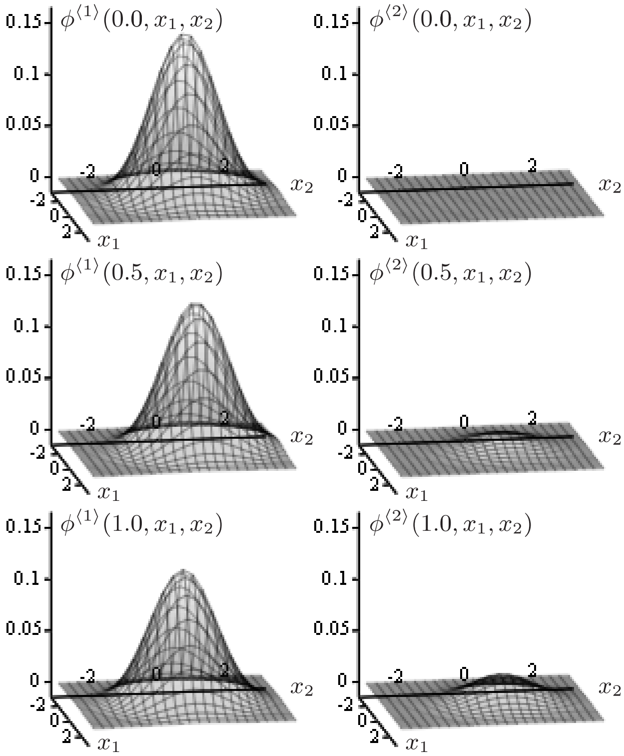

Fig. 1. Nonnormalized probability densities $\phi^{\langle 1\rangle}(t, x_1, x_2)$ and $\phi^{\langle 2\rangle}(t, x_1, x_2)$.

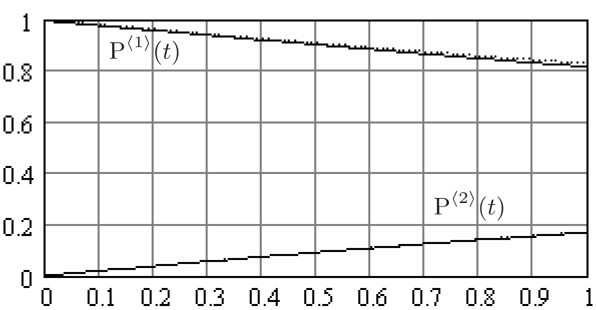

Fig. 2. Functions $\mathrm{P}^{\langle 1\rangle}(t)$ and $\mathrm{P}^{\langle 2\rangle}(t)$.

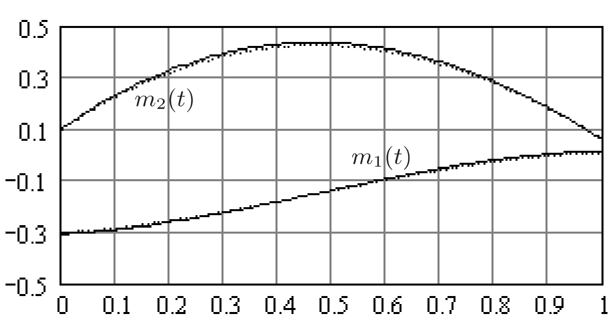

Fig. 3. Functions $m_1(t)$ and $m_2(t)$.

if $c_{i_1\ldots i_p k_1\ldots k_r j_1\ldots j_q l_1\ldots l_s} = a_{i_1\ldots i_p j_1\ldots j_q} b_{k_1\ldots k_r l_1\ldots l_s}$, $i_1,\ldots,i_p,k_1,\ldots,k_r,j_1,\ldots,j_q,l_1,\ldots,l_s = 0,1,2,\ldots$.

5. Let $A(p,q) = [a_{i_1\ldots i_p j_1\ldots j_q}]$ be an infinite $(p+q)$-dimensional matrix. An infinite $(q+p)$-dimensional matrix $B(q,p) = [b_{j_1\ldots j_q i_1\ldots i_p}]$ is said to be the transpose of $A(p,q)$ if $b_{j_1\ldots j_q i_1\ldots i_p} = a_{i_1\ldots i_p j_1\ldots j_q}$, $i_1,\ldots,i_p,j_1,\ldots,j_q = 0,1,2,\ldots$. We will use the notation $[A(p,q)]'$ to denote the transpose of $A(p,q)$.

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